A Modified Whiteness Test for Damage Detection Using Kalman Filter Innovations

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ABSTRACT

Deviations from whiteness in the innovations of a Kalman filter indicate that the filter is not optimal for the given data. Accepting that the disturbances are stationary and white lack of optimality derives from the fact that the values of some parameters have changed between the time the filter was formulated and the present. The parameters that define the filter come from system properties and from the statistics of the disturbances. For the filter to perform effectively as a fault detector it is necessary to ensure that deviations from whiteness are not due to variations in the statistics of the noise. This paper examines the mathematical relation between the covariance function of the innovations and the changes in the disturbance statistics. It is shown that the effect of changes in the noise statistics on the discriminating metric can be minimized by shifting the range of lags for which the metric is evaluated. The fact that the modified whiteness test can retain adequate sensitivity to damage is illustrated numerically.

INTRODUCTION

Most approaches to the damage detection problem can be classified as parameter estimation techniques or residual based schemes [1]. In the parameter estimation strategy the changes in estimated model parameters are used to decide whether or not significant changes have taken place. In the residual approach the likelihood of changes is judged by inspecting differences between measurements and reference signals obtained from a reference model and the measurements. A residual based approach to detection contains the following components: 1) a model of the reference condition, 2) a scheme to generate a residual 3) a metric derived from the residuals and 4) criteria for deciding whether the metric indicates change or no change.

The classical residual detector uses a whiteness test on the innovations of a Kalman filter [2] and was introduced by Mehra and Peschon [3]. A particularly attractive feature of the Kalman filter whiteness test is the fact that the method can be used when all inputs that drive the system are stochastic. On the negative side, however, the filter is such that the detection algorithm identifies not only changes in the system properties but also changes in the statistics of the disturbances. As one gathers, if the contribution to the discriminating metric coming from fluctuations in the disturbances are significant when compared to those due to damage then the resolution of the detection task deteriorates. This paper examines the mathematical relation between changes in the noise statistics and the covariance function of the innovations. Based on the results obtained a modified whiteness test that is insensitive to variability of the disturbances but retains sensitivity to damage is formulated. The merit of the modified test is illustrated in the context of simulations.

ANALYTICAL FORMULATION

System Considered

It is assumed that the system considered has a description in sampled time given by

$$x(k+1) = Ax(k) + G\omega(k) \tag{1}$$

$$y(k) = Cx(k) + v(k) \tag{2}$$

where A $\in R^{NxN}$ and C $\in R^{mxN}$ are the system and the state to output matrices, G $\in R^{Nxr}$ specifies the distribution of the disturbances, ω $\in R^{Nx1}$ and $v \in R^{mx1}$ are the disturbances and the measurement noise, y $\in R^{mx1}$ is the output vector and x $\in R^{Nx1}$ is the state vector. The integers, N, m and r are the system order, the number of outputs and the number of independent disturbances, respectively. It is assumed that ω and ν are Gaussian zero mean and white with covariance matrices Q and R.

Innovations

Assuming that the system matrices in the reference state are known and that the (steady state) Kalman filter gain K has been computed (to be discussed subsequently) the innovations e(k) are obtained from

$$\hat{x}^-(k+1) = A\hat{x}^+(k) \tag{3}$$

$$\hat{x}^+(k) = \hat{x}^-(k) + Ke(k) \tag{4}$$

$$e(k) = y(k) - C\hat{x}^-(k) \tag{5}$$

where e is the innovation sequence and the superscripts minus and plus indicate before and after the information from the measurement at the current time station is accounted for. The steady state Kalman gain is given by

$$K = APC^T \left[R + CPC^T \right]^{-1} \tag{6}$$

where P, the expected value of the state error, is the solution to the algebraic Riccati equation

$$P = APA^T - APC^T \left[R + CPC^T \right]^{-1} CPA^T + GQG^T \tag{7}$$

Data Driven Model

In the formulation outlined in Eqs3-7 the matrices {A,G,C,Q and R} have to be known. In practice the matrices Q and R are seldom known and it is appropriate, for detection purposes, to formulate the Kalman filter exclusively from measurements. A consequence of extracting the filter from the data is the fact that the physical significance of the state is lost. Nevertheless, for the purpose of computing the innovation sequence of Eq.5 this matter is immaterial. A data-driven Kalman filter model (for a second order MCK system) is a particular form of the ARMAX structure wherein the order of the autoregressive, exogenous and moving average terms are the same.

The Discriminating Metric

A test statistic that quantifies the "whiteness" of a signal can be defined in several ways [3]. Here we select the sum of the square of the covariance function of the signal (normalized to unit variance) for a preselected number of lags. The empirical estimate for the covariance of the normalized innovations $\tilde{e}$ is given by

$$\tilde{e} = H^{-0.5}(e_j - \overline{e}) \tag{8}$$

where

$$H = \frac{1}{L}\sum_{j=1}^{L}(e_j - \overline{e})(e_j - \overline{e})^T \tag{9}$$

the length of the sequence is L and $\overline{e}$ is the innovations mean. In Eq.8 the matrix square root is the unique non-negative option. The covariance function of the normalized innovations is, by definition

$$C_k = \frac{1}{L-k}\sum_{j=1}^{L-k}\tilde{e}_j \tilde{e}_{j+k}^T \tag{10}$$

for k = 1, 2, ... , p, where p is the number of lags considered. On the premise that $L \gg k$ all C_k, under the null hypothesis (i.e., that the system is undamaged) are identically distributed random variables with variance 1/L [3]. Normalizing the distribution to unit variance one has

$$\tilde{C}_k = \sqrt{L} \cdot C_k \tag{11}$$

For any entry in the diagonal of Eq.11 one can compute the test statistic q_p as

$$q_p = \sum_{k=1}^{p} \tilde{C}_{k,j}^2 \tag{12}$$

which follows a χ^2 distribution with *p* degrees of freedom. The probability that the value from Eq.12 in any given realization is larger than any given number is readily obtained from the *cdf* of the χ^2 distribution for the appropriate # of DOF. For example, if one selects the probability of being exceeded at 5% and takes *p*=20 the cutoff value is 31.4. In practice the approach sets the damage or no damage classification as a hypothesis test. The null hypothesis s accepted if the test statistic is smaller than the selected threshold and rejected otherwise. It is worth noting, however, that this procedure is not without difficulty because all the information one actually has is conditional on the fact that the system is undamaged and it does not (in principle) allow for objective statements regarding the probability of identifying damaged. Nevertheless, if the test statistic has a probability distribution for the aggregate of all the possible damage scenarios of interest that is strongly shifted to the right relative to the reference then this standard operating mode is reasonable.

Effect of Fluctuations in the Noise Covariance Matrices

When the noise statistics change the Kalman filter formulated for the reference state becomes an arbitrary observer for which one has

$$C_0 = CPC^T + R \tag{13}$$

$$C_j = C\bar{A}^j PC^T - C\bar{A}^{(j-1)} AKR \tag{14}$$

where P is the solution of

$$P = \bar{A}P\bar{A}^T + GQG^T + AKRK^T A^T \tag{15}$$

with

$$\bar{A} = A(I - KC) \tag{16}$$

Eqs.14 and 15 can be combined to obtain

$$vec(C_j) = E_{QQ} vec(Q) + E_{RR} vec(R) \tag{17}$$

where

$$E_{QQ} = (C \otimes C\bar{A}^j) N^{-1} E_Q \tag{18}$$

$$E_{RR} = [(C \otimes C\bar{A}^j) N^{-1} E_R - (I \otimes C\bar{A}^{(j-1)} AK)] \tag{19}$$

with

$$E_Q = G \otimes G \tag{20}$$

$$E_R = (AK \otimes AK) \tag{21}$$

and

$$N = I - (\bar{A} \otimes \bar{A}) \tag{22}$$

Details of the derivations of Eqs.13-22 are not presented here due to space limitations but the theory from where they can be derived can be found in [4,5]. Eq.17 shows that the covariance function of the arbitrary observer is linearly related to the vectorized forms of Q and R. From this result one gathers that the manner in which the chances in the noise affect the covariance function is strongly dependent on the spatial distribution of the disturbance fluctuations. In particular, changes that in vectorized form have strong projections in the direction of the vectors associated with the higher singular values of E_{QQ} and E_{RR} induce large correlations while those that project on vectors associated with the smaller singular values behave opposite. The well-known fact that multiplying Q and R by the same scalar does not affect the Kalman filter gain is made evident by noting that if the covariance function is zero this operation does not change it.

Modified Whiteness Test

Inspection of Eqs.18 and 19 show that the matrices E_{QQ} and E_{RR} involve the matrix $\bar{A}$ raised to powers that increase with the lag. Since this matrix has a spectral radius that is less than unity (i.e., the filter is stable) the entries decrease as the lags increase and one gathers that, for sufficiently large lags the changes in the disturbances will have no perceptible effect on the covariance function. A metric based on a modified Eq.12, namely

$$q_p = \sum_{k=p1}^{p2} \tilde{C}_{k,j}^2 \tag{23}$$

where the first lag is taken as p1 instead of one, and p = p2-p1+1, will have a χ^2 distribution that is essentially independent of the variations in the statistics of Q and R, provided p1 is large enough. If the correlation introduced by damage persisted at all lags p1 could be selected arbitrarily (provided it is small compared to L) but numerical results suggest that this is not the case. Namely, the sensitivity to damage of the metric of Eq.23 also decreases with lags, albeit at a different rate than that due to variations in Q and R. One way to choose p1 is by inspecting the eigenvalues of the matrix in Eq.16 and selecting a value such that the largest eigenvalue in absolute value, raised to p1, is smaller than some pre-selected number (0.1 for example). Work on the optimal selection of p1 and on the number of lags on which to base the test statistic is part of our present focus.

Effect of Damage in the Covariance Function

Ideally one would like to have an analytic expression that shows how damage affects the covariance of the innovations but derivation of such an expression has proven difficult because the innovations in the damaged condition cannot be viewed as sequences that would be obtained from a suboptimal filter connected to the reference system. One possibility that appears to hold some promise is to formulate the damage in terms of pseudo-forces but we have not at this point carried out this examination.

Numerical Examination

Consider a 5-DOF lumped-mass system with springs connecting mass #1 to #2, #2 to #3, etc, plus two additional springs (k_6 and k_7), connecting mass #1 to #3 and mass #2 to #4, respectively. All springs have a constant of 100 and the masses equal 0.05 in some consistent set of units. The damping distribution is classical with 2% damping in each of the 5 flexible modes and rigid body motion is restricted by grounding spring #1. To simplify the presentation we consider only one sensor consisting of an acceleration measurement on the 5th mass.

We take the disturbances as a realization from a covariance matrix that varies from one simulation to the next. The bounds that specify the entries in Q are selected assuming that the uncertainty on the spatial distribution is significantly less than that on the mean intensity. Specifically, we take the diagonal of Q in each simulation as a vector whose entries are realizations from a uniform distribution with limits of 0.75 and 1.50 and subsequently scale the complete diagonal matrix by a constant, also selected from a uniform distribution, but in this case with limits of 0.25 and 4. Damage is simulated as 10% loss of stiffness on spring#1. The results are presented in Fig.1 in the form of a Receiver Operating Characteristic, or ROC curve. The curve is derived using two hundred simulations of the healthy state and 200 hundred simulations of the damaged condition. The ROC curve illustrates the tradeoff that is made between false positives and false negatives when a discriminating threshold is selected. Although several formats are used the most common is the one used in Fig.1, wherein the horizontal axis is the false positive rate and the vertical is one minus the rate of false negatives which, of course, is the rate of true positive detection. As can be seen, the lag shifted test displays has the potential of offering a reasonably good performance while in the standard test no threshold can lead to reasonable performance because the ROC curve deviates little from the case of no information given by a straight line that goes from the origin to {1,1}.

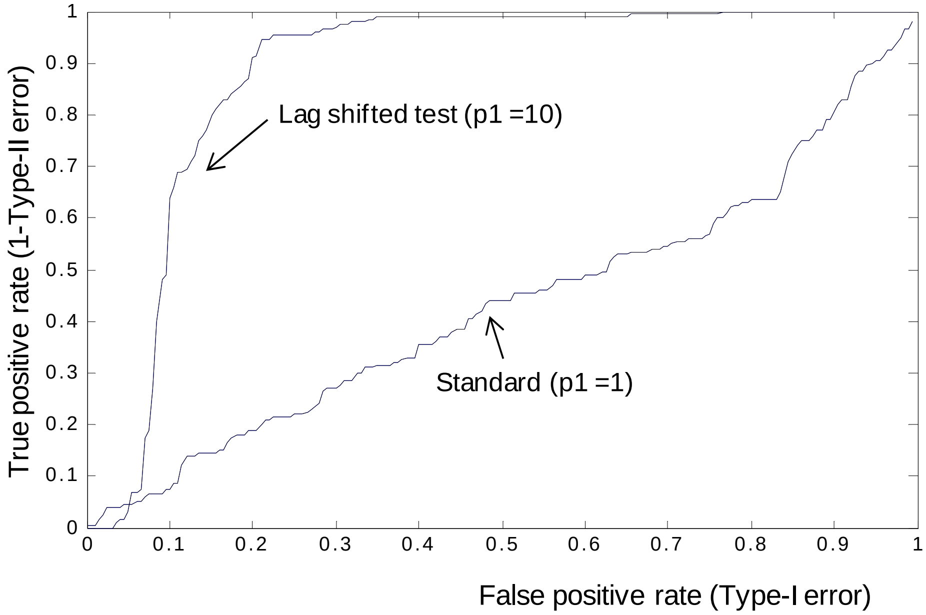

Figure 1. ROC curve illustrating the gains realized by the shifted whiteness test; damage considered is 10% loss of stiffness in spring #1.

The optimal threshold in any given case is a function of the ROC curve and of the relative cost of making a type-I or a type-II error. For illustration we assumed that the cost of a false negative (type-II error) is one unit and that of a type-I is β units and compute the total cost for β = 0.5, 1 and 2. The results are plotted in Fig.2 and illustrate that the optimal threshold cut (for these cost ratios) is insensitive to β and is connected with a type-I error of around 0.2 (examination shows that the metric in this region is about 38).

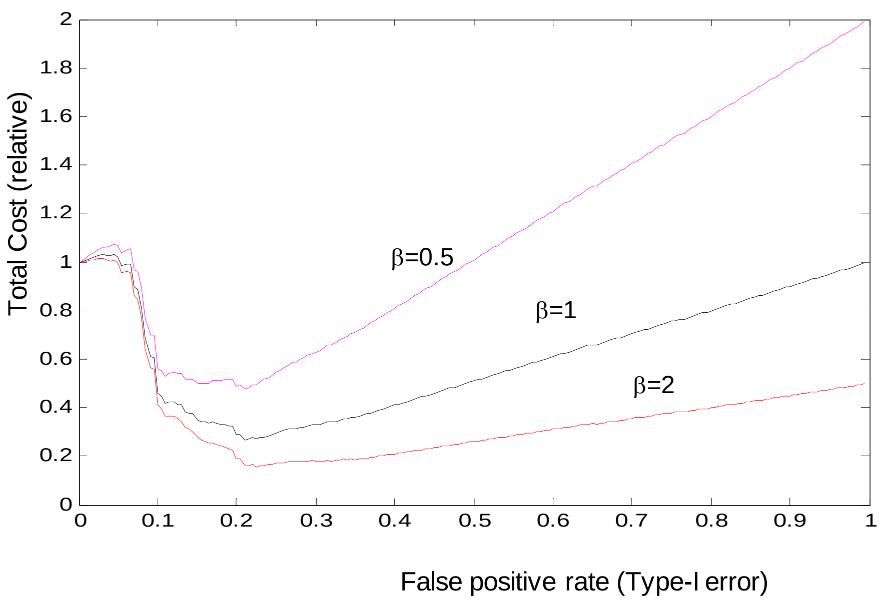

Figure 2. Total cost vs Type-I error rate for the ROC curve of Fig.1

It is opportune to note that in actual applications one typically does not have information about the damage scenarios and the metric threshold cut has to be selected based on information about the behavior of the type-I error.

CONCLUDING REMARKS

Use of a Kalman filter as a fault detector hinges on the fact that when the filter is operating optimally the innovation signals are white. In the real situation the whiteness property implies that the entries of the covariance function of the finite length innovations are realizations from a Gaussian distribution with zero mean and known variance and this allows one to cast the whiteness question as a hypothesis test at some selected level of confidence. The issue addressed in this paper is the fact that lack of optimality in the filter arises not only when the properties of the system change but also when the statistics of the disturbances vary. The basic contribution of the paper is the observation that the correlations resulting from changes in the noise model can be minimized by shifting the whiteness test to higher lags and that this can be done without removing the sensitivity of the correlations to damage.

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